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V. *The Principal Constituent of the Tides of the North Sea.*

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1. *Introduction.*

This paper is concerned exclusively with the principal lunar semi-diurnal harmonic constituent of the tides which is denoted by $\mathbf{M}_2$. The primary object is to show how the fundamental dynamical equations of the tides may be used to obtain a knowledge of the distribution of the surface-elevation from such observational data as are available. The dynamical equations, as formulated in § 4, connect the elevation-gradients with the currents and the external forces, including those of friction. From a knowledge of the currents and a hypothesis for the frictional forces the elevation-gradients can be calculated.

When the elevation is also known the directions of the co-tidal and co-range lines, and also the distance apart of neighbouring members of these lines, can be calculated. Such conditions are fulfilled for coastal stations, and it is remarkable that, in spite of the great attention that has been paid to co-tidal charts, these simple calculations do not appear to have been previously made. But if the elevation-gradients can be calculated along a line which passes through one or more points at which the elevation is known, it is clear that methods can be devised by which the elevation can be calculated all along the line. Again, such calculations do not appear to have been previously made.

In 1913 J. P. Jacobson considered the calculation of the tidal elevations in the Kattegat from observations on currents, but instead of using the fundamental differential equations he used the relations for a simple purely progressive frictionless wave in one dimension in the absence of the Earth's rotation. In 1918 G. I. Taylor, in considering the same problem for the tides of the Irish Channel between Arklow and Bardsey Island,

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* §§ 1–4 by J. Proudman, being part of the Adams Prize Essay, 1923; §§ 5–13 by A. T. Doodson. The authors are much indebted to Miss A. L. Cooper and Miss S. K. Lowry, of the Tidal Institute, for computational assistance. The work was done while the Institute was in receipt of grants from the Department of Scientific and Industrial Research. The chief results of the paper were given in a Lecture at the meeting of the British Association at Hull in 1922

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used two simple "Kelvin-waves"* freely progressing in opposite directions. In 1920 R. O. Street made calculations of a converse nature relating to the tides of the Irish Sea. Assuming a knowledge of the elevations he used the differential equations of frictionless motion to calculate the currents, but decided that it was only practicable to evaluate the sum of the maximum and minimum speeds at each point considered. Some of Street's data are of questionable accuracy and his process is one of numerical differentiation. Such a process even with relatively good data is an uncertain one.

Before proceeding to the main purpose of the paper it is of interest to recall previous work and speculation on the tides of the open North Sea, and to give, as far as possible, a qualitative explanation of these tides on dynamical principles. This is done in the next two sections. The notation is explained in § 4.

2. *Previous work on North Sea Tides.*

In 1833 W. Whewell in his 'Essay towards a first approximation to a Map of Co-tidal Lines,' included a chart proposing positions for these lines near the coasts of the North Sea. He was led to perceive the existence of the amphidromic point in the Flemish Bight, and in 1836 he published the chart shown in fig. 1. This perception led to the observational verification by W. Hewett (1841) of the small range near the point, confirmation of which has been obtained by the pressure-gauge of L. Favé (1910).

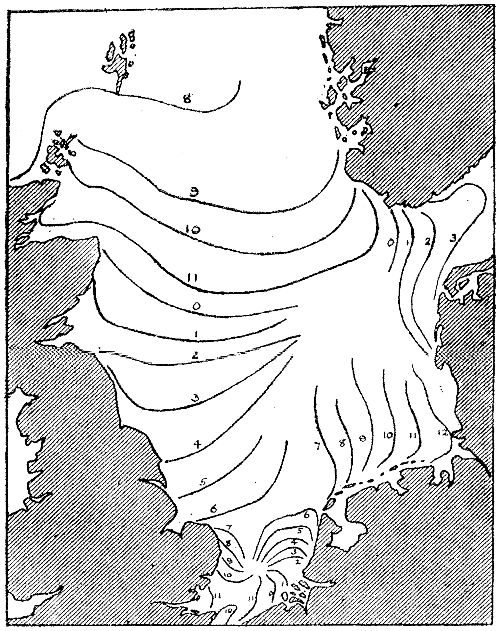

Fig. 1.—Co-tidal lines according to Whewell.

The numbers give approximately in mean lunar hours the times of high water after the transit of the moon at Greenwich.

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* Waves of the type which can be propagated in a rotating channel, with no transverse current.

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But G. B. Airy, in his 'Tides and Waves' of 1845, pronounced against the existence of this amphidromic point, and expressed the view that the distribution of tides over the North Sea was very largely determined by the distribution of depth.

In 1904 R. A. Harris gave the dynamical explanation of amphidromic points of what we may call the "narrow sea type," a type which includes that of the Flemish Bight. In the same work he proposed the set of co-tidal lines shown in fig. 2; they involve also a coastal amphidromic point round which the phase-lag increases by about 540°.

In 1905 J. P. van der Stok showed that the direction of rotation of the currents off the Dutch coast was opposite to that indicated by Airy's rule,* and in 1910 that the progression there did not consist of a purely progressive wave.

In 1918 A. Defant applied the "narrow sea" dynamical theory (in which transverse currents are neglected) to the tides of the Flemish Bight, associating them with those of the English Channel. In his step-by-step integration of the differential equations he allowed for friction by introducing a term proportional to the speed of the current.

In 1920 R. Sterneck proposed the set of co-tidal lines shown in fig. 3, which is very much nearer the results of the present paper (see fig. 17, p. 209) than any previously given. Sterneck briefly indicated the principles underlying the construction of his chart as being founded on those of the narrow-sea theory. In the same year G. I. Taylor

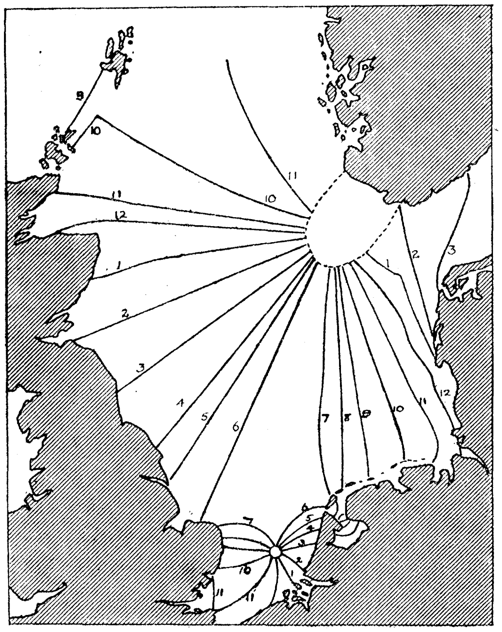

Fig. 2.—Co-tidal lines according to Harris.

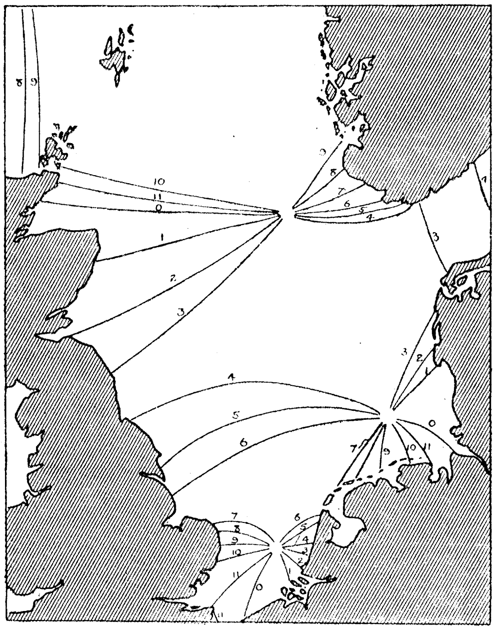

Fig. 3.—Co-tidal lines according to Sterneck.

The numbers give approximately in mean lunar hours the times of high water after the transit of the moon at Greenwich.

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* 1845, § 363 (see references on p. 218).

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gave the mathematical theory of the reflection of a Kelvin-wave at the head of a rotating rectangular gulf of uniform depth, and in order to illustrate his results he took a basin corresponding as nearly as possible to the North Sea. In 1923 A. DEFANT, quoting TAYLOR's results, pointed out some of the defects of the narrow-sea theory as explaining the tides of the North Sea.

In a second paper of the same year A. DEFANT took the navigational information relating to the tidal currents and found a satisfactory degree of consistency with the dynamical equations. He then used the continuity equation to obtain the elevations from the same data, and thus produced co-tidal and co-range charts for the North Sea. These charts are in general agreement with that of STERNECK, but as regards method, they mark a great advance. They are also in general agreement with fig. 17 of the present paper, but in points of detail they are considered inferior to this for the following reasons :—

1. DEFANT's data appear to be much inferior.
2. He does not use coastal elevations in the determination of open-sea elevations, but relies entirely on the rough current data.
3. His processes are essentially those of differentiation, whereas those of the present paper are essentially integrations.

Of course the fact noted as No. 2 is a strong argument in favour of the applicability of the fundamental equations used both by DEFANT and in the present paper.

3. *Qualitative explanation of the North Sea Tides.*

The tides of any body of water are dynamically determinate when either the elevations or the normal currents are known along the sections of the channels which connect the body with other waters. But the numerical details of the necessary calculations have been carried out, up to the present, only for such natural basins as have very small transverse currents.

We suppose that we have given the M_2 currents normal to two bounding sections, and find it convenient to take these bounding sections through the two outer amphidromic points shown in fig. 3, though we do not assume the amphidromic nature of these points. We assume from observation that across the northern section the current everywhere reaches a southerly maximum at $\sigma t = 0$ and across the southern section a northerly maximum at $\sigma t = 90°$.

A simple calculation shows that the effect of the local astronomical forces is negligible, so that the tides are entirely maintained by the currents just mentioned.

We first examine the possible standing oscillation maintained by the currents across the northern section only, in the absence of the earth's rotation. This is shown in figs. 4 and 5, in which the currents and elevation-gradients respectively are at a maximum.

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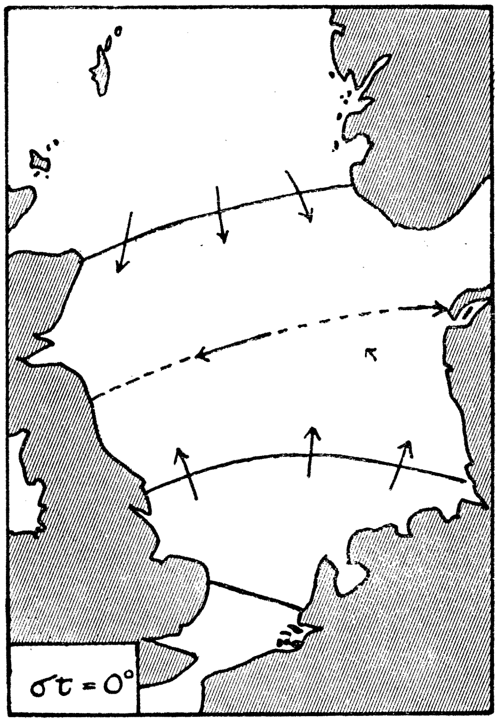

Fig. 4.

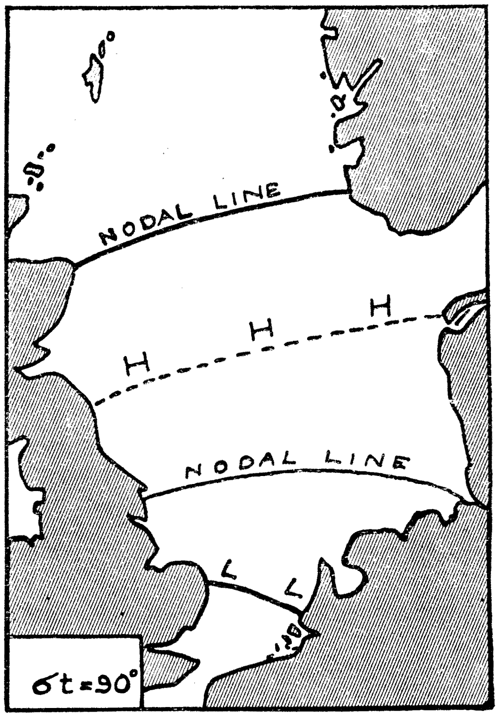

Fig. 5.

H, L denote high and low water respectively.

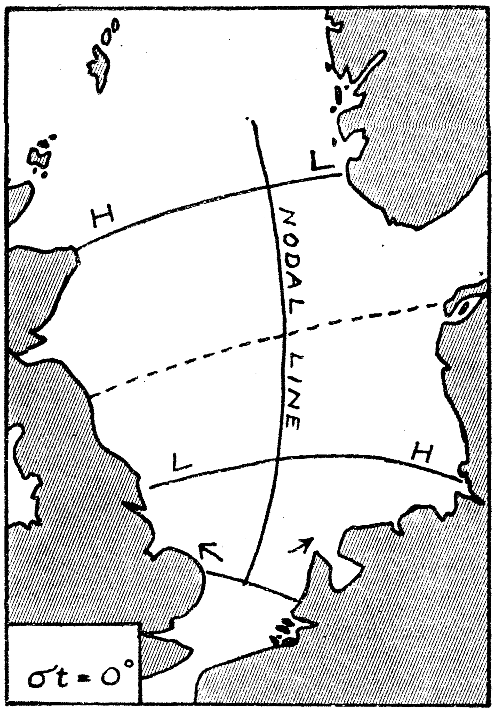

Fig. 6.

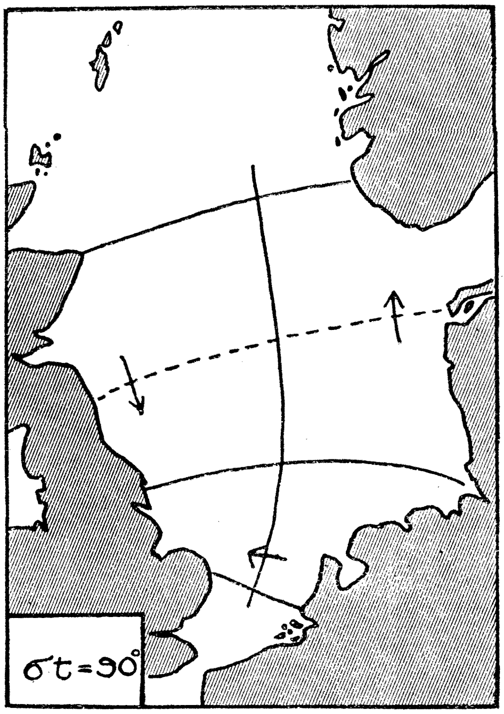

Fig. 7

H, L denote high and low water respectively.

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The positions of the nodes and loop depend on the distribution of depth, and in fact the northern section proves to be a nodal line. The currents along and near the loop are obtained by applying an extension of AIRY's rule, to the effect that a quarter of a period before high water at any place the currents there are directed towards the shoals. But the complexity of the distribution of depth makes the application of this rule rather uncertain.

We next examine the effect of the earth's rotation; from TAYLOR's work we are led to suppose the results to be as indicated in figs. 6 and 7, which are to be superposed on figs. 4 and 5 respectively. The tides may now be regarded as consisting of a wave travelling round the sea in the positive direction and representing a progression of energy. The effect of friction will be the continual dissipation of this energy, so that the amplitudes on the east coast will be smaller than those on the west coast and this effect will become more pronounced towards the north. It follows that the amphidromic points will be displaced towards the east, the northerly one suffering a greater displacement than the central one. In this way we obtain the co-tidal lines shown in fig. 8.

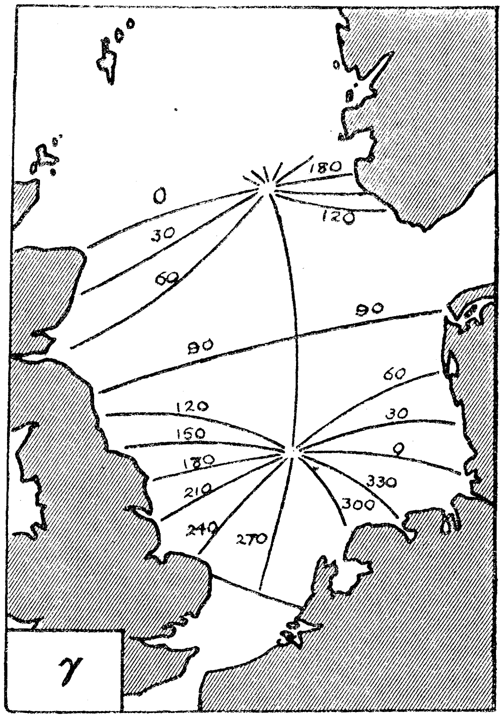

Fig. 8.—Co-tidal lines in absence of currents at southern section; the numbers give, in degrees, the values of γ.

We now introduce the currents across the southern section and assume that they are stronger than those we have considered so far. Due to these currents alone we should

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have an amphidromic point of the narrow sea type (figs. 9 and 10). This point remains in the presence of the tides of fig. 8, being displaced a little towards the Dutch coast.

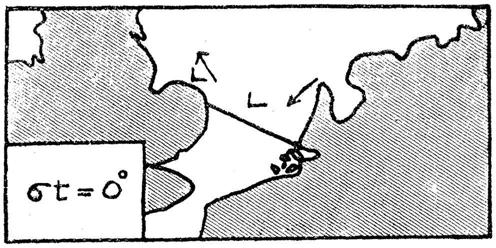

Fig. 9.

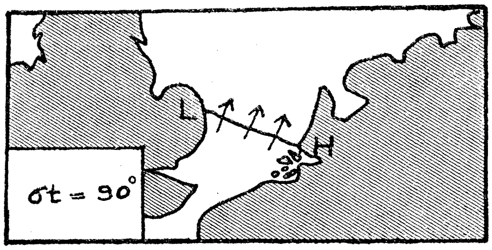

Fig. 10.

H, L denote high and low water respectively.

From fig. 9 we see that to the north of the Flemish Bight we shall have $\gamma = 180°$, and if we superpose this on the tides of fig. 8 we shall obtain a value of γ lying between 180° and 270°. As a matter of fact, owing to the stronger currents across the southern section, the value of γ in the mouth of the Flemish Bight is only a little greater than 180°, and the superposition gives the set of co-tidal lines of fig. 11. This is a very good approximation to that given in fig. 17.

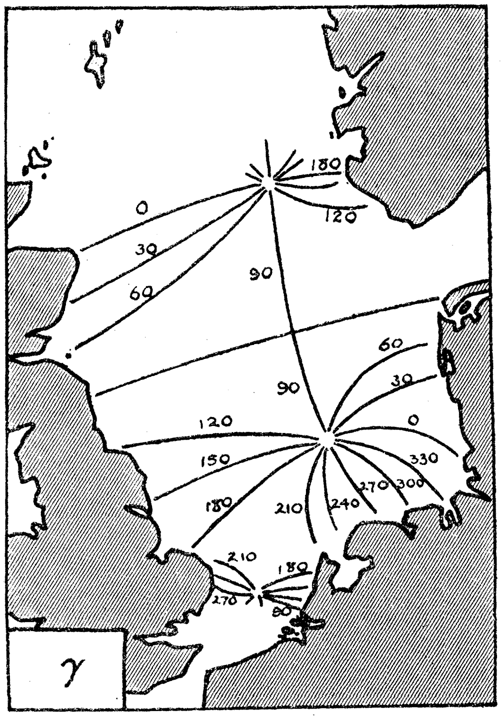

Fig. 11.—Co-tidal lines. The numbers give, in degrees, the values of γ; for comparison with figs. 1, 2, 3, γ should be divided by 30.

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The chief part of the precise current-information obtained from observation is shown in fig. 12, and it is interesting to compare the currents at the individual stations with those of the foregoing theory.

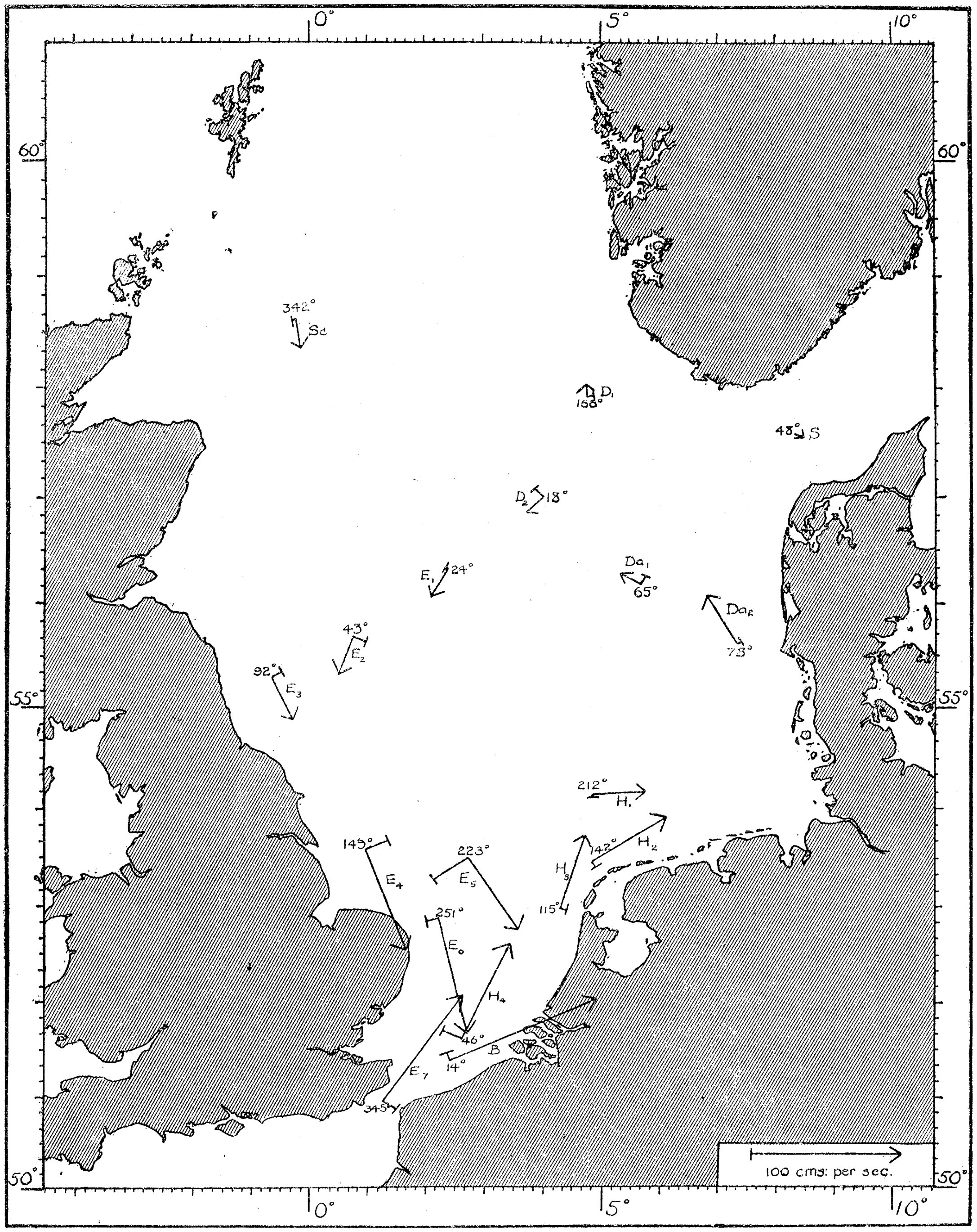

Fig. 12.—Maximum flood and following minimum currents, denoted respectively by ——→ and ——|, the station being at the common point of these two vectors. The numbers give, in degrees, the values of γ'

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Stations Sc and D_1 are both near the northern bounding section, so that the fact of their currents reaching a southerly maximum about $\sigma t = 0$ has been assumed (fig. 4). The practical absence of rotation at Sc is in accordance with our theory.

Stations E_3, E_2, E_1 and D_2 lie near the loop of fig. 5 and at $\sigma t = 0$ their currents are directed towards the English coast, thus agreeing with fig. 4. At $\sigma t = 90°$ E_3 has a strong southerly current, E_2 a weaker southerly current and E_1 practically zero current, while D_2 has a small northerly current. These observational facts agree well with fig. 7.

Station Da_1 has a westerly current at $\sigma t = 0$ and a northerly current at $\sigma t = 90°$, which facts agree well with figs. 4 and 7 respectively.

Station Da_2 has a flood parallel to the shore at $\sigma t = 75°$, which is in good agreement with fig. 11, and the behaviour of a progressive wave.

Station E_4 has a flood parallel to the shore at $\sigma t = 150°$ agreeing with fig. 11, and a current towards the shore a quarter period before high water, in accordance with AIRY's rule.

Station E_5 has an easterly current at $\sigma t = 180°$ and a northerly current at $\sigma t = 90°$ in agreement with figs. 6 and 10 respectively.

Station E_6 only differs from E_5 in the relative strengths of its component currents.

Stations H_3 and H_2 have maximum currents decidedly earlier than $\sigma t = 210°$ and 240°, their times of high water respectively, which may be accounted for by their nearness to the strong currents of the southern section at $\sigma t = 90°$.

Station H_1 has an easterly flood at $\sigma t = 210°$ in agreement with fig. 11 and the behaviour of a progressive wave.

It is interesting to notice that we have accounted for negative current-rotations at Stations D_2, E_5 and E_6, whereas TAYLOR's result, corresponding to uniform depth and to the absence of normal currents at the southern bounding section, would give positive rotations for these stations.

4. *Fundamental Equations.*

We shall denote by :—

Ω the angular speed of the earth's rotation on its axis ;
g the acceleration of gravity ;
ρ the density of the water, supposed constant and uniform ;
λ, L the north latitude and west longitude respectively of any point of the sea ;
x, y Cartesian co-ordinates of position in the mean surface of the sea ;
h the depth of the water below any point of the mean surface ;
t the time ;
ζ the elevation of the free surface of the water at any time, above any point of the mean surface ;
u, v, ν the mean values along any vertical of the velocity-components at any point and time in the horizontal directions of increasing x, y and the normal to a section respectively ;

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ρF, ρG the components of the frictional force per unit area of the sea-bottom ;
σ the "speed" of the harmonic motion, *i.e.*, 2π/period ;
H the amplitude (semi-range) of the elevation at any point ;
γ the lag in phase of the elevation behind that of the corresponding equilibrium constituent on the meridian of Greenwich ;
u_0, v_0 the maximum and minimum speeds respectively of the current at any place ;
γ' the phase, at the instant of maximum flood current, of the elevation in the corresponding equilibrium constituent on the meridian of Greenwich ;
ψ, ψ' the angles by which the co-tidal and co-range lines respectively at any place are in advance of the direction of x-increasing.

We define the "flood" to be that maximum current which occurs within a quarter period of high water, either before or after, or else exactly a quarter period before. We take u_0 to be positive and the sign of v_0 the same as that of rotation of the current according to the mathematical convention.

We also take

$$\left.\begin{aligned} \zeta &= \zeta_1 \cos \sigma t + \zeta_2 \sin \sigma t \\ u &= u_1 \cos \sigma t + u_2 \sin \sigma t \\ v &= v_1 \cos \sigma t + v_2 \sin \sigma t \\ F &= F_1 \cos \sigma t + F_2 \sin \sigma t \\ G &= G_1 \cos \sigma t + G_2 \sin \sigma t \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots (1)$$

where all the functions with suffixes depend on position but not on time.

For a *co-tidal line* we have

$$\frac{\zeta_2}{\zeta_1} = \text{constant}, \qquad \tan\psi = -\frac{\zeta_1 \dfrac{\partial \zeta_2}{\partial x} - \zeta_2 \dfrac{\partial \zeta_1}{\partial x}}{\zeta_1 \dfrac{\partial \zeta_2}{\partial y} - \zeta_2 \dfrac{\partial \zeta_1}{\partial y}}, \quad \ldots \ldots \ldots (2)$$

and, if $\partial/\partial s$ denotes differentiation in a direction perpendicular to the co-tidal line,

$$H^2 \frac{\partial \gamma}{\partial s} = \left\{\left(\zeta_1 \frac{\partial \zeta_2}{\partial x} - \zeta_2 \frac{\partial \zeta_1}{\partial x}\right)^2 + \left(\zeta_1 \frac{\partial \zeta_2}{\partial y} - \zeta_2 \frac{\partial \zeta_1}{\partial y}\right)^2\right\}^{\frac{1}{2}} . \quad \ldots \ldots \ldots (3)$$

For a *co-range line* we have

$$H^2 = \zeta_1{}^2 + \zeta_2{}^2 = \text{constant}, \qquad \tan\psi' = -\frac{\zeta_1 \dfrac{\partial \zeta_1}{\partial x} + \zeta_2 \dfrac{\partial \zeta_2}{\partial x}}{\zeta_1 \dfrac{\partial \zeta_1}{\partial y} + \zeta_2 \dfrac{\partial \zeta_2}{\partial y}}, \quad \ldots \ldots (4)$$

and, if $\partial/\partial s$ denotes differentiation in a direction perpendicular to a co-range line,

$$H \frac{\partial H}{\partial s} = \left\{\left(\zeta_1 \frac{\partial \zeta_1}{\partial x} + \zeta_2 \frac{\partial \zeta_2}{\partial x}\right)^2 + \left(\zeta_1 \frac{\partial \zeta_1}{\partial y} + \zeta_2 \frac{\partial \zeta_2}{\partial y}\right)^2\right\}^{\frac{1}{2}} . \quad \ldots \ldots \ldots (5)$$

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On neglecting the effect of the local astronomical disturbing forces, the dynamical equations of motion may be written in the form

$$\left.\begin{aligned}\frac{\partial u}{\partial t}-2\omega v&=-g\frac{\partial\zeta}{\partial x}-\frac{F}{h}\\ \frac{\partial v}{\partial t}+2\omega u&=-g\frac{\partial\zeta}{\partial y}-\frac{G}{h}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(6)$$

where

$$\omega=\Omega\sin\lambda.\quad\ldots\ldots\ldots\ldots\quad(7)$$

Following G. I. TAYLOR (1918) and others, we shall assume that the magnitude of the frictional force is proportional to the square of the speed of the current, so that

F, G

are the appropriate harmonic constituents of

$$ku\,(u^2+v^2)^{\frac{1}{2}},\qquad kv\,(u^2+v^2)^{\frac{1}{2}},\quad\ldots\ldots\ldots\ldots\quad(8)$$

k being a constant.

On substituting from (1) into (6) we obtain

$$\left.\begin{aligned}\sigma u_2-2\omega v_1&=-g\frac{\partial\zeta_1}{\partial x}-\frac{F_1}{h}\\ -\sigma u_1-2\omega v_2&=-g\frac{\partial\zeta_2}{\partial x}-\frac{F_2}{h}\\ \sigma v_2+2\omega u_1&=-g\frac{\partial\zeta_1}{\partial y}-\frac{G_1}{h}\\ -\sigma v_1+2\omega u_2&=-g\frac{\partial\zeta_2}{\partial y}-\frac{G_2}{h}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(9)$$

If we measure time from the instant of equilibrium high water on the meridian of Greenwich, we have

$$\zeta=H\cos(\sigma t-\gamma),\quad\ldots\ldots\ldots\ldots\quad(10)$$

$$\frac{\zeta_2}{\zeta_1}=\tan\gamma,\quad\ldots\ldots\ldots\ldots\quad(11)$$

and high water at any place occurs when $\sigma t=\gamma$.

When a single station only is under consideration it is convenient to take the x-axis in the direction of the flood, and to measure time from the instant of flood, so that

$$\left.\begin{gathered}\frac{\zeta_2}{\zeta_1}=\tan(\gamma-\gamma'),\\ u_1=u_0,\qquad u_2=v_1=0,\qquad v_2=v_0,\\ F_2=G_1=0.\end{gathered}\right\}\quad\ldots\ldots\ldots\quad(12)$$

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The equations (9) then give

$$\left.\begin{aligned} g\frac{\partial\zeta_1}{\partial x} &= -\frac{F_1}{h} \\ g\frac{\partial\zeta_2}{\partial x} &= \sigma u_0 + 2\omega v_0 \\ g\frac{\partial\zeta_1}{\partial y} &= -\sigma v_0 - 2\omega u_0 \\ g\frac{\partial\zeta_2}{\partial y} &= -\frac{G_2}{h} \end{aligned}\right\} , \quad \ldots \ldots \ldots \ldots \quad (13)$$

and on substituting into (2) and (4) we obtain respectively

$$\tan\psi = -\frac{(\sigma u_0 + 2\omega v_0) + \tan(\gamma - \gamma')\, F_1/h}{\tan(\gamma - \gamma')\,(\sigma v_0 + 2\omega u_0) - G_2/h}. \quad \ldots \ldots \ldots \quad (14)$$

$$\tan\psi' = \frac{\tan(\gamma - \gamma')\,(\sigma u_0 + 2\omega v_0) - F_1/h}{(\sigma v_0 + 2\omega u_0) + \tan(\gamma - \gamma')\, G_2/h} \quad \ldots \ldots \ldots \quad (15)$$

The integral

$$g\rho\int h\nu\zeta\, ds \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (16)$$

taken along the trace of any vertical section represents the rate of doing work across the section, the positive direction being that of measurement of the current. If the line of this integral has l, m for the direction cosines of its normal, the mean value of the integrand of (16) becomes

$$\tfrac{1}{2}g\rho h\mathrm{H}\,\{lu_0\cos(\gamma - \gamma') + mv_0\sin(\gamma - \gamma')\} \quad \ldots \ldots \ldots \quad (17)$$

This we may regard as the normal component of a vector whose components parallel to the axes are

$$\tfrac{1}{2}g\rho h\mathrm{H}u_0\cos(\gamma - \gamma'), \qquad \tfrac{1}{2}g\rho h\mathrm{H}v_0\sin(\gamma - \gamma'), \quad \ldots \ldots \ldots \quad (18)$$

and such a vector we shall call the "mean flow of energy vector." We notice that the first component of (18) is positive, so that the direction of the mean flow of energy makes an acute angle with the direction of flood.

5. *Data.*

The most valuable tidal information from which to construct charts is that which is obtained from a long series of observations and reduced by the harmonic method. This ideal, however, is not attainable at present and the information relating to tidal heights and currents falls into four grades :—

Grade 1 : observations extending over a period not less than a year and reduced according to the harmonic method.

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Grade 2: observations extending over a much shorter period (perhaps only a fortnight) but reduced according to the harmonic method.

Grade 3: observations of unstated duration expressed by non-harmonic constants.

Grade 4: isolated observations, extending over only a day or two, and unreduced.

For many stations on the coasts of the North Sea and for a few light-vessels off those coasts there are available the harmonic constants for M_2. These are given in Table I (p. 211) and this information is substantially in the first grade. The places for which the constants are available are, however, somewhat unevenly scattered. There are practically no elevation constants of Grades 1 or 2 for places out at sea.

The information available for currents is substantially of Grade 2, long series of observations at sea being difficult to obtain. The principal current observations taken in the North Sea and reduced according to the harmonic method have been taken under the auspices of "Le Conseil Permanent International pour l'Exploration de la Mer," and these provide the data given in Table II. The stations at which observations were taken will be referred to as "international stations." In some cases the observations have been repeated in other years, but in no one case are more than six weeks of observations available. These data are illustrated in fig. 12.

The information given in Tables I and II provides the framework of the Charts to be given later, but use has had to be made of information of Grades 3 and 4.

The Admiralty Tide Tables (Part II) provide plenty of Grade 3 information for elevations; the data so available consist of the values of the high-water interval at full and change of the moon, together with the average spring range. This would need reduction in some way or other to fit it for our use, but the 'United States Coast and Geodetic Survey Tables' give the mean high-water interval (H.W.I.) and the mean range of tide. In the absence of shallow water, a long series of observations so reduced would give quantities readily transformed into harmonic constants. The mean high-water interval, multiplied by the speed of the M_2 constituent, would give κ, while half the mean range of tide would give H. We require γ, which exceeds κ by twice the west longitude. Accepting for the present the reductions given in the 'U.S.A. Tide Tables,' we deduce H and γ and regard these temporarily as the true harmonic constants. These data are given in Table III.

Some use has been made of a number of Grade 4 observations of tidal elevations at sea. Many of these have been obtained from the Hydrographic Department of the Admiralty and have not been published. They are most abundant near to and in the Flemish Bight. It is not an easy matter to obtain the semidiurnal high water from the observations, because of the prominence of quarter- and sixth-diurnal tides. A convenient method of treating these has been by harmonic analysis of twelve consecutive hourly heights so chosen as to give the semidiurnal high water about the middle of the sequence. The crude "constants" so obtained give the semi-range and time of high water of the semidiurnal tide fairly well. To reduce approximately to M_2 constants

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it has then been assumed that the range at a given station bears always a constant ratio to the range at a standard port, such as Hull or Dover, and the times of H.W. at the "station" and "port" differ always by the same interval. This is quite probably a reasonable assumption to make, if we may judge from the harmonic constants for M_2 and S_2 at various places in the North Sea; in any case it is the best hypothesis available. If the range-ratio and interval between high waters is determined from observations we can assume them applicable to the M_2 constituent. Since the harmonic constants at the "port" are known, then approximate values of the M_2 harmonic constants at the "station" can be obtained; a list of such approximations is given in Table IVA.

Other Grade 4 observations of elevations published by HOLZHAUER (1886) and quoted by KRUMMEL (1911) are given in Table IVB, and some observations published by MERZ (1921) are also given in Table IVC.

Some Grade 3 and 4 information relating to currents in the North Sea is found in the Admiralty publication 'Tides and Tidal Streams'; this gives the directions, spring-speeds and time of turning relatively to high water at Dover.

Some unpublished current-information (Grade 4) has been available, but has only been used on one or two occasions.

6. *Coastal Directions.*

The Admiralty manual of 'Tides and Tidal Streams' (1909) contains much information concerning currents along the coasts, only a little of which, however, has been utilised in the present work. The fundamental formulæ giving the outward directions of the co-tidal line and co-range line at a given station on the coast ($v = 0$) become :—

$$\tan \psi = -\frac{\sigma}{2\omega} \cot (\gamma - \gamma')$$

$$\tan \psi' = +\frac{\sigma}{2\omega} \tan (\gamma - \gamma')$$

if friction is ignored; it is sufficient to take $\sigma/2\omega = 1 \cdot 2$ throughout the North Sea. The value of γ' has been defined as the phase-lag of maximum flood stream, but it is convenient here to use it as the phase-lag of maximum stream, to the south on the English Coast and to the north on the continental coast. The results utilised are as follows :—

	ψ	ψ'
	°	°
Girdleness to Rattray Head	126	46
Off St. Abb's Head	71	—26
Off Coquet Islands	90	0
Off Whitby	103	18
Flamborough Head to Humber	123	43

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	ψ °	ψ' °
Off Blakeney Overfalls	115	34
Off Cromer	100	14
Off Yarmouth	135	55
Outside the River Ems	100	14
Blaavand Point to Nyminde Gab	50	−51
Thyborön	90	0

Though friction has been ignored in the above computations, it is a simple matter to show from (14) and (15) that the effect of friction is to rotate both co-tidal and co-range line to the right of the stream (*i.e.*, in the negative direction), and a little allowance has to be made for this effect in using the above results.

7. *Calculation of Elevation-Gradients.*

We take the co-ordinate axes of x and y to the south and to the east respectively, and the time origin we fix as the time of passage of the mean moon at Greenwich. Then the first procedure is to evaluate u_1, u_2, v_1, v_2 from the data of Table II., and these are given in Table V. for all the international stations; where more than one set of observations have been recorded the average is taken.

Now by definition, with $U = \sqrt{u^2 + v^2}$,*

$$F_1 = \text{co-efficient of } \cos \sigma t \text{ in } kUu.$$
$$F_2 = \quad ,, \quad \sin \sigma t \text{ in } kUu.$$
$$G_1 = \quad ,, \quad \cos \sigma t \text{ in } kUv.$$
$$G_2 = \quad ,, \quad \sin \sigma t \text{ in } kUv.$$

Taking $\sigma t = 0°, 30°, 60° \ldots 330°$ then u, v, U, Uu, Uv were constructed consecutively, for each single station. The values of Uu, Uv were analysed by the method of least squares and the co-efficients of $\cos \sigma t$, $\sin \sigma t$ in each so obtained. The factor k being taken as $0 \cdot 002$, we obtained for each station the frictional data given in Table VI.

We have now, in c.g.s. units,

$$\left.\begin{aligned} 10^7 \frac{\partial \zeta_1}{\partial x} &= -au_2 + bv_1 - cF_1 \\ 10^7 \frac{\partial \zeta_1}{\partial y} &= -av_2 - bu_1 - cG_1 \\ 10^7 \frac{\partial \zeta_2}{\partial x} &= \quad au_1 + bv_2 - cF_2 \\ 10^7 \frac{\partial \zeta_2}{\partial y} &= \quad av_1 - bu_2 - cG_2 \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (19)$$

---

* Here u, v refer to the total current and not to its M_2 constituent, but the fractional error made in neglecting other constituents is small and has been committed. *Cf.* 'Brit. Ass. Report,' 1923, p. 303.

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where

$$\left.\begin{aligned} a &= 10^7\sigma/g \quad = 1\cdot432 \\ b &= 10^7 2\omega/g = 1\cdot487 \sin\lambda \\ c &= 10^7/gh \quad = 1\cdot019 \div h/10000 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

the numerical values following from

$$\left.\begin{aligned} \sigma \quad &= 28^\circ\cdot9841 \text{ per mean solar hour} \\ 2\Omega &= 30^\circ\cdot081 \text{ per mean solar hour} \\ g \quad &= 981 \text{ cm.}/(\text{sec})^2 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

The values of the gradients are given in Table VII. for each of the international stations.

8. *Calculation of Elevations.*

Two methods have been used for the evaluation of ζ_1 and ζ_2 at a selection of points at sea. The first method is one of numerical integration, using equations of type—

$$\delta\zeta = \frac{\partial\zeta}{\partial x}\delta x + \frac{\partial\zeta}{\partial y}\delta y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

where the values of the gradients are taken at the middle of the interval δx, δy.

The second method is one of graphical interpolation. Expressed in general terms for two variables ξ, η, the direct plotting of two or three values of η against ξ is supplemented by a knowledge of two or three gradients $(d\eta/d\xi)$ at the same or at intermediate points; the curve of η against ξ is then drawn as accurately as possible to fit all the conditions.

Each method supposes that the stations supplying data lie on a smooth curve on the map, and fortunately, several of the international stations lie on lines whose curvature is small; these are denoted by

Line I: through stations E_3, E_2, E_1, D_2.
Line II: through stations E_3, E_2, E_1, D_1.
Line III: through stations Sc, D_2, Da_1, Da_2.

Other lines used chiefly for interpolation purposes are defined by

Line IV: through H_2, H_1, and along the meridian, $L = -4^\circ\cdot87$.
Line V: through E_2 and along the parallel of latitude $\lambda = 55^\circ\cdot67$.
Line VI: through H_3, E_5, E_4 to Spurn Head.
Line VII: a straight line through Sk, S, and cutting Line I in $L = -2^\circ$; III in $L = -4^\circ\cdot52$.

The lines are illustrated in fig. 13.

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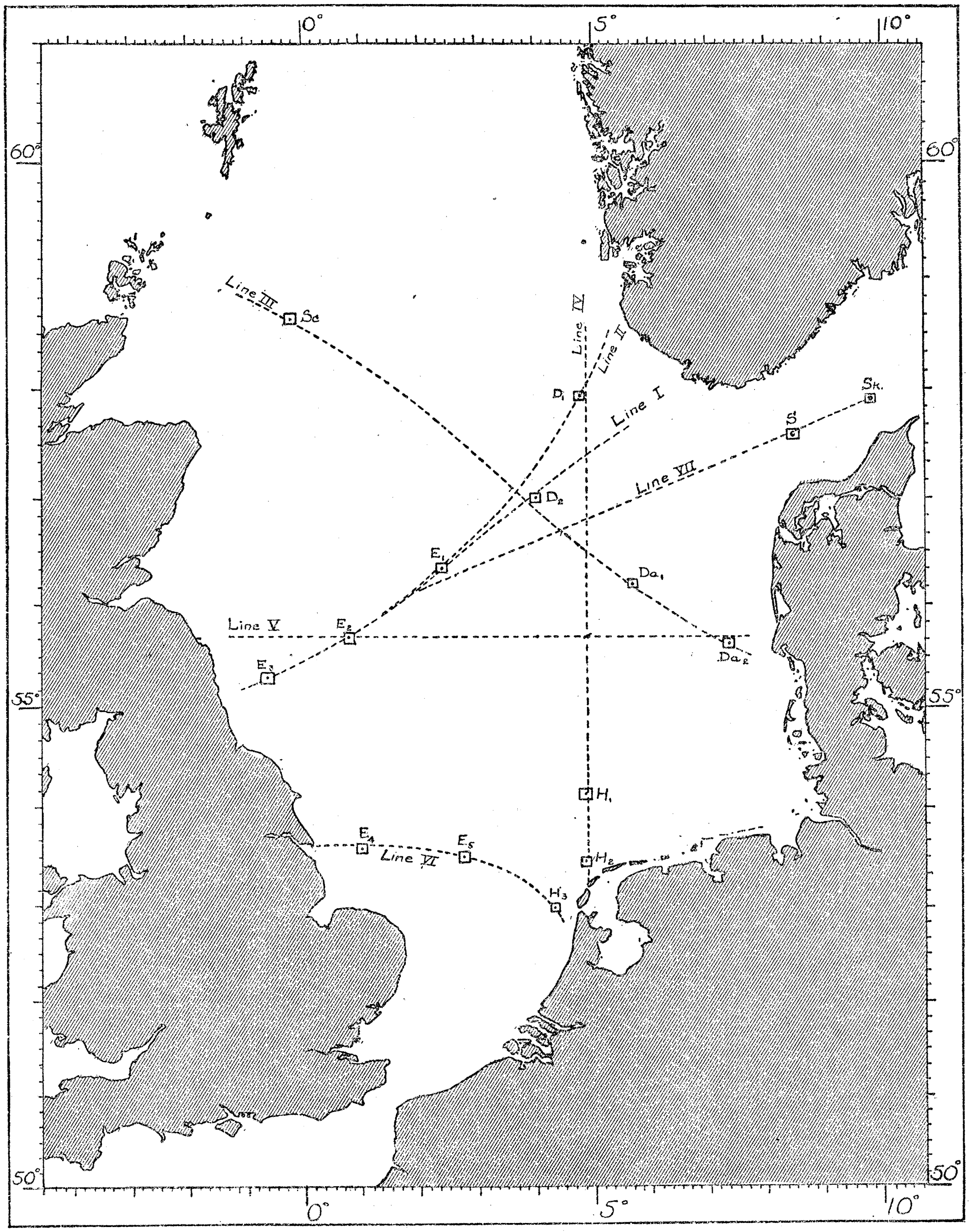

Fig. 13.—Sectional lines.

The integration method requires gradients at a number of points at small intervals along the line, together with the values of ζ_1, ζ_2 at one point on it, and the numerical equation between x and y for the line. The accuracy of this method depends upon the accuracy of the interpolated values of the gradients; interpolation is unsatisfactory with less than four stations and also if there is much variability in the gradients at these

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primary stations. The graphical interpolation method, however, requires a knowledge of ζ_1, ζ_2 at each end of the line; it could not have been used for Lines I to III for the following reasons. A glance at fig. 13 shows that Lines I and II, produced, cut the coasts of England and of Norway, while Line III terminates on the coast of Denmark and among the Orkney Islands. Now for none of the stations on these lines do we know ζ_1, ζ_2, and we have to rely entirely on coastal values; but these are not readily obtainable except on the English Coast for Lines I and II, where there is an approximately straight coast line free from sandbanks. The tides off S. Norway are so small that exact tidal information has not been obtained; the tides among the Orkney Islands are so complex and subject to local diffraction effects that exact values of ζ_1, ζ_2 at the northern end of Line III cannot be obtained, while this line cuts the coast of Denmark among a maze of sandbanks. An error of 10 cm. in ζ_1, or ζ_2 may mean a considerable misplacement of an amphidromic point. The integration process has been the only recourse for Lines I to III, and it is fortunate that the number of stations is sufficiently large and the variability in gradients sufficiently small to give a high degree of definiteness to the interpolated values required for use with small intervals.

The curvature of Lines I to III being small the gradients $\partial\zeta_1/\partial x, \ldots$ were plotted in each case against L and read off at the middle points of the chosen intervals of 0°·2 in L; the corresponding values of $\delta\lambda$ being obtained from the numerical relation between λ and L for the line, we have

$$\left.\begin{aligned} \delta x &= -(\delta\lambda)^\circ \times 1\cdot112 \times 10^7 \\ \delta y &= -(\delta \mathrm{L})^\circ \times 1\cdot112 \times 10^7 \times \cos\lambda \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (23)$$

and the values of $\delta\zeta_1$, $\delta\zeta_2$ follow from formula (22). Summing these from some arbitrary origin on Line I gave $\zeta_1 - \mathrm{C}_1$, $\zeta_2 - \mathrm{C}_2$; C_1, C_2 being constants. For Lines II and III the arbitrary origins are at E_1, D_2 respectively, and the values of the integration constants were chosen so as to give continuity with Line I. The values of ζ_1, ζ_2 on the English Coast determined C_1, C_2.

Assuming for the present that the integration constants are satisfactorily known, then we proceed to consider Lines IV and V together. These were most readily dealt with by the graphical method, since ζ_1 and ζ_2 were known from Table I. at H_1 and H_2 on Line IV, and the gradients at these stations were known from Table VII., while ζ_1, ζ_2 and their gradients were known at the points where this line cuts Lines I to III. For Line V we knew ζ_1, ζ_2 and their gradients at two points, one on Line I and one on Line III. Now Lines IV and V must yield identical values of ζ_1 and ζ_2 at their point of intersection and this was a valuable aid to drawing the curves. The curves of ζ_1 for each line were drawn on one sheet and the abscissæ λ and L, for Lines IV and V respectively, were so arranged that the abscissæ $\lambda = 55°\cdot67$, $\mathrm{L} = -4°\cdot52$ were superposed; then the ζ_1 curves must have a common ordinate for this common abscissa. In drawing these curves it was most convenient to deal with $\partial\zeta/\partial\lambda$ and $\partial\zeta/\partial\mathrm{L}$ instead

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of $\partial\zeta/\partial x$, $\partial\zeta/\partial y$, the transposition being effected by (23). Line VI was dealt with by the graphical method, partly* because only three international stations lie on the line and these are not quite sufficient to give interpolated values of sufficient exactness for the integration method. The values of ζ_1, ζ_2 at Spurn Point have been obtained from Table III., but for reasons similar to those explained later (§ 10) the amplitude has been reduced from 223 cm. to 210 cm., the value of $\gamma = 153°$ being accepted as substantially correct. The values of ζ_1, ζ_2 at H_3 are obtained from Table I.

Line VII was also dealt with by the graphical method; ζ_1, ζ_2 and their gradients were (supposedly) known at the points where the line cuts Lines I and III and the gradients were known at the two stations S and Sk in the Skagerrack. For the latter stations, however, the values of ζ_1, ζ_2 had to be obtained from the coastal information. We now proceed to discuss the tides in the Skagerrack and then to discuss the best values of C_1, C_2 for Lines I to III.

9. *The Skagerrack.*

The tidal regime in and near the Skagerrack is of great interest theoretically, though the actual tidal elevations and currents are small. No true harmonic constants for elevations are available between Esbjerg and Hirtshals on the Danish Coast and between Arendal and Stavanger on the Norwegian Coast. The presence of two amphidromic points (see § 3 and figs. 3, 17) near the entrance to the Skagerrack causes some little difficulty in deducing approximate harmonic constants from Grade III data, but some guide is given from the times of H. W. F. C. as given in the Admiralty Tables; multiplying these by σ and adding twice the West Longitude gives the following :—

	°		°		°
Thyborön . . .	89	Langösund . . .	105	Stavanger . . .	275
Hirtshals . . .	97	Jomfruland . . .	97	Bergen	292
Skagen	141	Risör	95		
		Arendal	98		
		Oxö	97		
		Lindesnaes . . .	61		

while Table I. gives $\gamma = 96°$, $82°$, $271°$, $287°$ for Hirtshals, Arendal, Stavanger and Bergen respectively. The values of γ are increasing northwards along the North Sea

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* This region is much afflicted by sandbanks, so that the variations in depth bear a large proportion to the depth itself. In a canal with parallel walls we should expect the rate of flux across a transverse section to vary more steadily along the canal than the mean velocity in the section. Since the gradients are linear functions of the velocity-components it may in this case be questioned whether we are justified in expecting smoothness in the gradients. It may be stated here that the integration process beginning at H_3 gave the amplitude at Spurn Point about 40 cm. too great, while the use of gradients obtained from interpolations in $h\partial\zeta/\partial x$ ... gave an amplitude too small.

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coast of Norway and eastwards along the Skagerrack, and there is obviously a minimum $< 90°$ about Lindesnaes. At the stations Sk and S the value of γ must be about 90°, and the following table gives ψ and ψ' for $\gamma = 80°$ and 100° according to (2) and (4) :—

	Sk.	S.	Sk.	S.
°	°	°	°	°
$\gamma = 80$	$\psi = -19$	$\psi = -2$	$\psi' = -92$	$\psi' = -73$
$\gamma = 100$	$\psi = 5$	$\psi = 16$	$\psi' = -73$	$\psi' = -52$

Approximately, therefore, the co-tidal lines at these stations run north-south, and the co-range lines run east-west.

The values of the amplitudes at Arendal and Hirtshals are respectively 8 and 11 cm. The data given above are shown in fig. 14 and we deduce that at station S :—

$$H = 12, \qquad \gamma < 90°, \qquad \zeta_1 = 1, \qquad \zeta_2 = 12,$$

and at station Sk :—

$$H = 11, \qquad \gamma > 90°, \qquad \zeta_1 = -1, \qquad \zeta_2 = 11,$$

and that the co-tidal line $\gamma = 90°$ should be drawn as indicated in fig. 17.

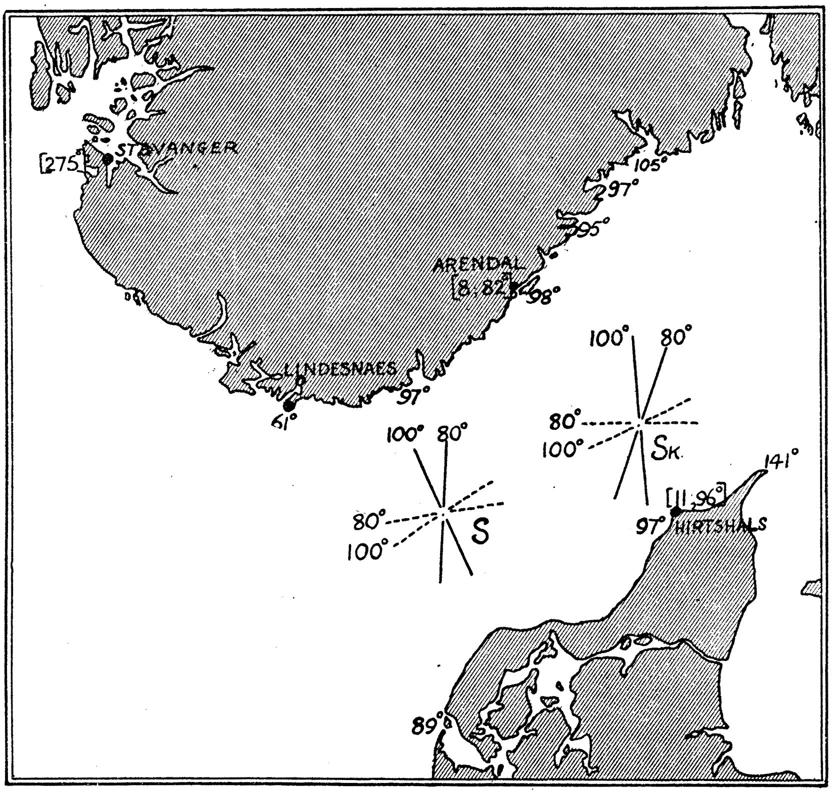

Fig. 14.—Data for the Skagerrack. Full lines are co-tidal lines ; broken lines are co-range lines, for assumed values of γ.

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10. *Determination of Integration-constants.*

Line I strikes the English Coast near Blyth, and interpolation from Table III. gives at this point $H = 173$, $\gamma = 94°$, $\zeta_1 = 173$, $\zeta_2 = 12$. The only harmonic constants available from Table I. in this region are for Hartlepool and Leith, and a comparison of the two tables shows that H in Table I is from 5 to 9 cm. less than H in Table III. while the difference in γ varies from $+4°$ to $-9°$. It must be remembered that Table III. gives the mean high-water interval and that this includes the shallow water effects. Some independent reductions of the data for H. W. F. C. and spring tides as given in the 'British Tide Tables' tend to show that we must take H a little less than the approximation given in Table III. and that the values of γ there given are substantially correct. An independent test of accuracy is furnished by the connection with Line VII, for in the region near to the Skagerrack the gradients are so small that an error of a few centimetres in any part of the line is readily detected. It was found that the uncorrected data of Table III gave a value of C_1 correct to within 2 centimetres, but that the corresponding value of C_2 should be decreased by about 10 cm. Combining these with the previous conclusions it is probable that the best values of C_1, C_2 are given by taking $H = 163$, $\gamma = 94°$ at the English end of Line I. The errors of the integration process for the Lines I to III are small (§ 8), and probably the total error in integrating from the English Coast to the intersection of Lines III and VII is less than 2 cm.

11. *The Flemish Bight.*

In this region, south of Line VI, there is little difficulty in drawing co-tidal and co-range lines, for the coastal data are very good, many of the stations in Table I. being located here. Also stations H_3 and H_4, in the open sea, are among those for which elevation constants are given in Table I. For station B lying near to H_4 we can evaluate harmonic constants from

$$\delta\zeta = \frac{\partial\zeta}{\partial x}\,\delta x + \frac{\partial\zeta}{\partial y}\,\delta y$$

where δx, δy refer to the interval between the two stations and the average gradients in the interval are used. This method gives $H = 150$, $\gamma = 358°$ at station B.

Further, a good part of Table IVA refers to this region and though the entries in that table may not be individually trustworthy, yet as a whole they are consistent with an amphidromic point about Latitude 52° 32′ N., Longitude 2° 54′ E. There is very little doubt about the positions of the co-tidal lines and co-range lines, but this is resolved by making use of the current data at stations E'_6, E_7, B, H_4. These stations are not suitably placed for either of the processes used with sectional lines, but use has been made of equations (2) and (4). For stations H_3, H_4 and B the elevation constant γ is known and the slopes (ψ and ψ') of the co-tidal lines and of the co-range lines through these places can be accurately computed. For other stations ψ and ψ' were evaluated

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for two assumed values of γ in each case, a little interpolation and redrawing of the co-tidal lines previously sketched then giving the correct slopes pertaining to the γ finally indicated by the lines. The values of ψ and ψ' are as follows :—

Station.	γ °	ψ °	ψ' °
H_3	171	113	41
H_4	7	21	282
B	358	16	270
E'_6	210	252	147
	240	281	174
E_7	310	15	267
	330	40	258

12. *Directions of co-tidal lines and co-range lines.*

These are readily evaluated from (2) and (4). In the case of Lines I to III the values of the gradients and of ζ_1 and ζ_2 have been obtained at small intervals along the line. For other lines only three stations have been available, and in some cases graphical interpolation of gradients has not been sufficiently accurate for the integration process for ζ_1 and ζ_2, but the results are usually accurate enough for the present purposes. In some cases, however, the values of the gradients have been deduced from the values of ζ_1 and ζ_2 as obtained direct by the graphical interpolation process.

In determining the values of ψ and ψ' from the tangent formula the angles have been chosen for which $\psi + 90°$, $\psi' + 90°$ give the directions in which γ and H are respectively increasing.

13. *Construction of Charts.*

Table VIII. contains a summary of the information obtained along the sectional Lines I to VI, while figs. 15 and 16 exhibit the results of § 6, Table VIII., and the original data of Tables I., III. and IV. Both these figures indicate clearly the existence of three amphidromic points and it is a comparatively simple matter to sketch the co-tidal lines and co-range lines shown in fig. 17. In figs. 15 and 16 the directions for which γ and H are increasing are indicated by arrows on the lines giving the directions of the co-tidal and co-range lines.

The information yielded by Line VII has chiefly been of value in connection with the arbitrary constants in the integration process for Lines I to III; it only remains to add that the co-tidal line for $\gamma = 90°$ cannot cross Line VII except at a point well in the Skagerrack. Also no permissible modification of Line III can be made which allows a co-tidal line for $\gamma = 90°$ to cross Line III and to run thence to the amphidromic point off the coast of Norway. There are two places on Line VI for which $\gamma = 180°$,

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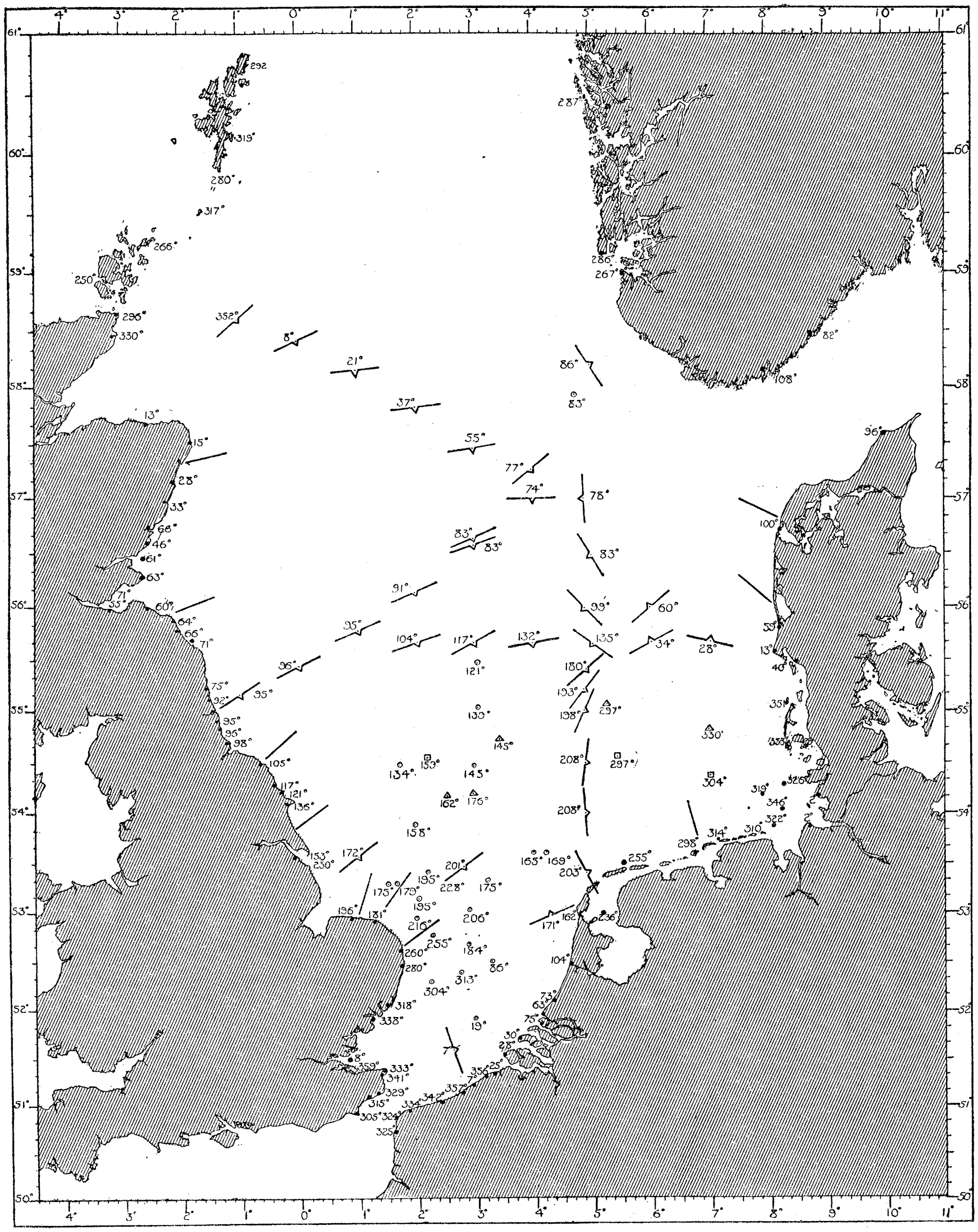

Fig. 15.—Values of γ and directions of co-tidal lines.

with a small possibility of these being on one line running from the English Coast, circling round to near the Flemish Coast, and thence to the amphidromic point in the Flemish Bight; such a possibility is suggested chiefly by some of the Grade IV. information, but tests for the gradients, using some unpublished current information, indicate that the co-tidal line for $\gamma = 180°$ commencing on the English Coast must run to the

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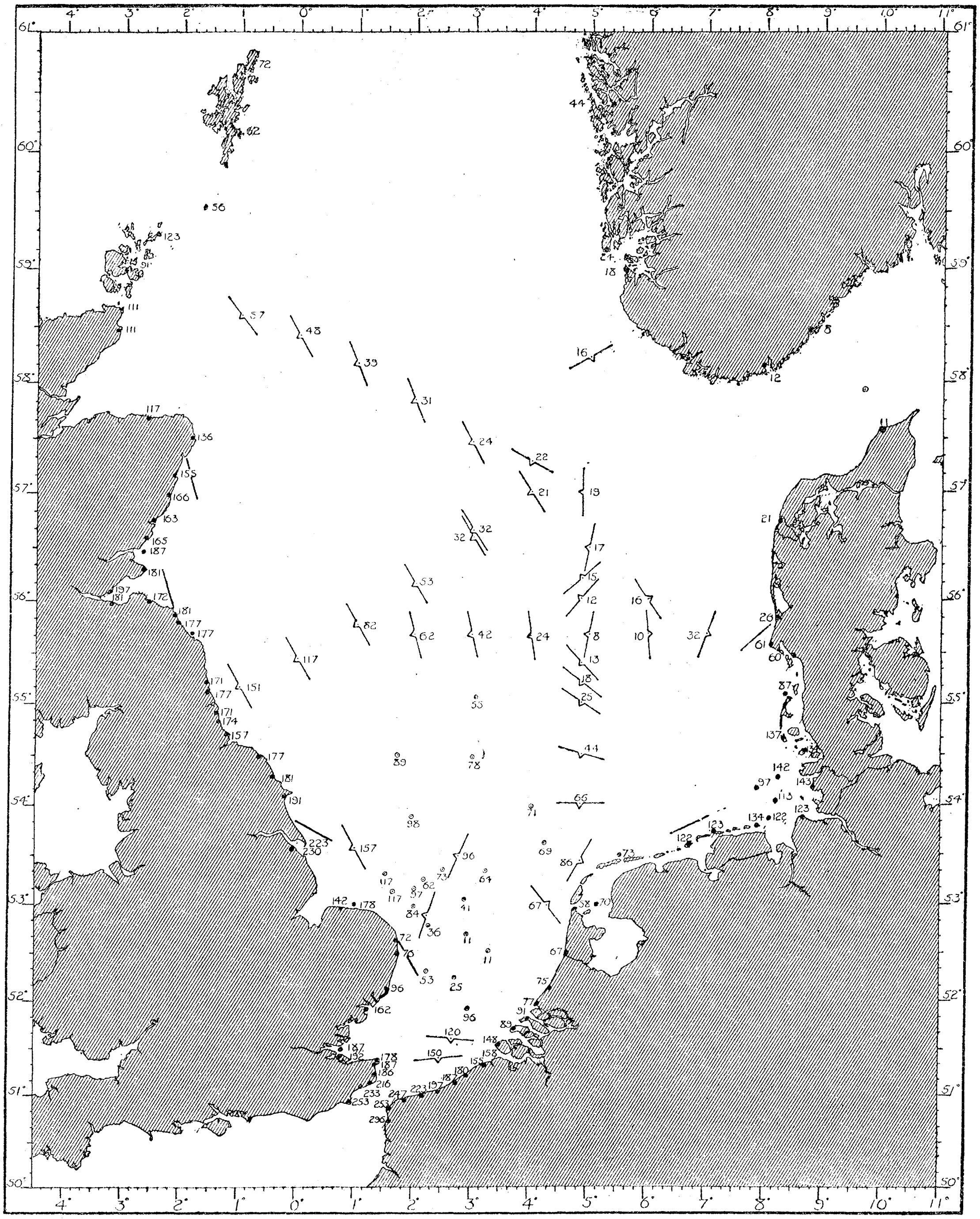

Fig. 16.—Values of H and directions of co-range lines.

middle amphidromic point, and consequently a line for $\gamma = 180°$ must run from the lower amphidromic point to the Flemish Coast.

Line VI also gives the information that no co-range line for $H < 66$ crosses this section.

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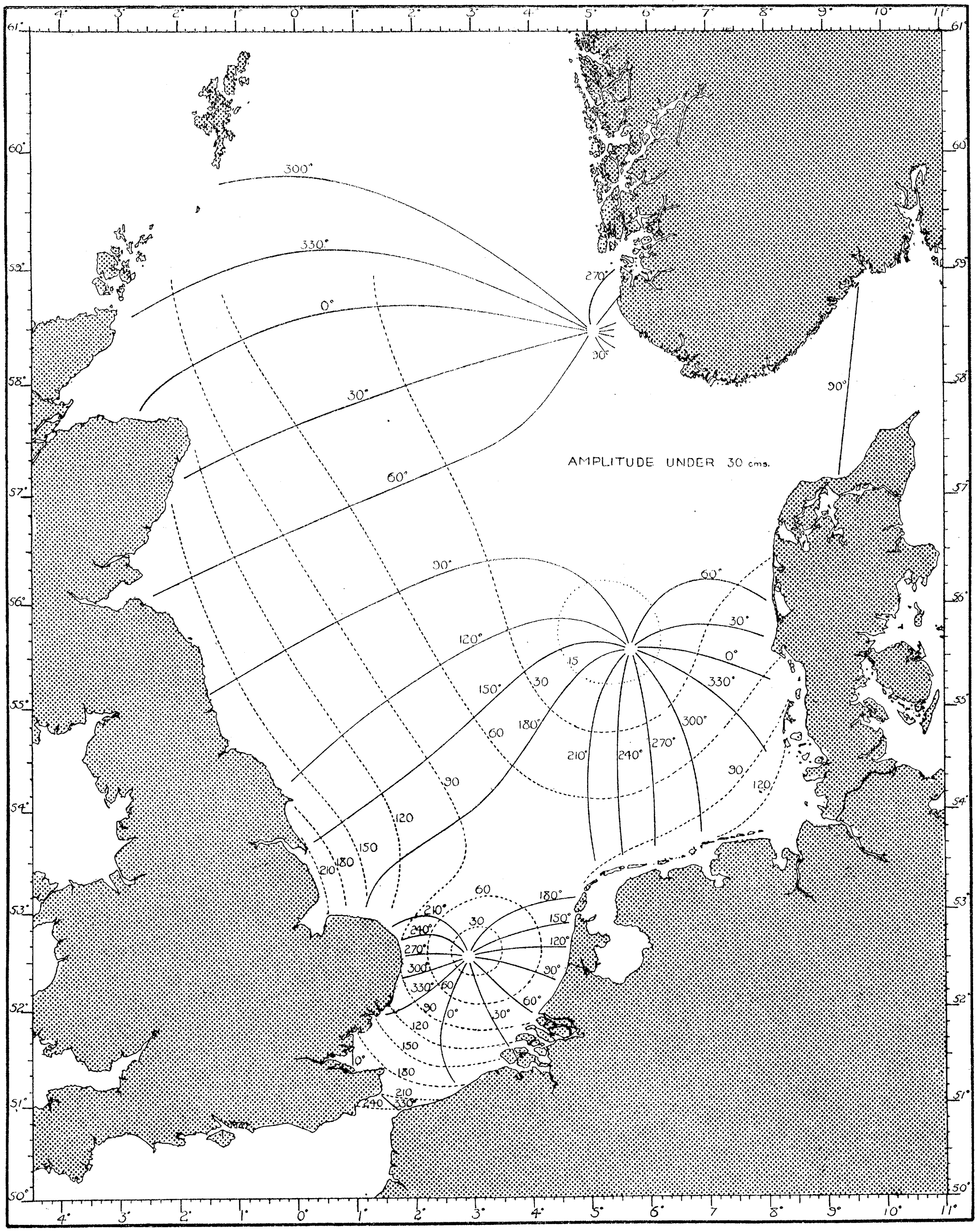

Fig. 17.—Co-tidal lines and co-range lines, denoted by full and dotted lines respectively. The associated numbers give the values of γ in degrees and of H in centimetres.

The details round the northern amphidromic point are a little uncertain, but there is no doubt that an amphidromic point must be very near to the coast of Norway, and that γ has a minimum value somewhere near Lindesnaes. In constructing the Chart the coastal data and outward gradients of co-tidal lines have been used, special weight

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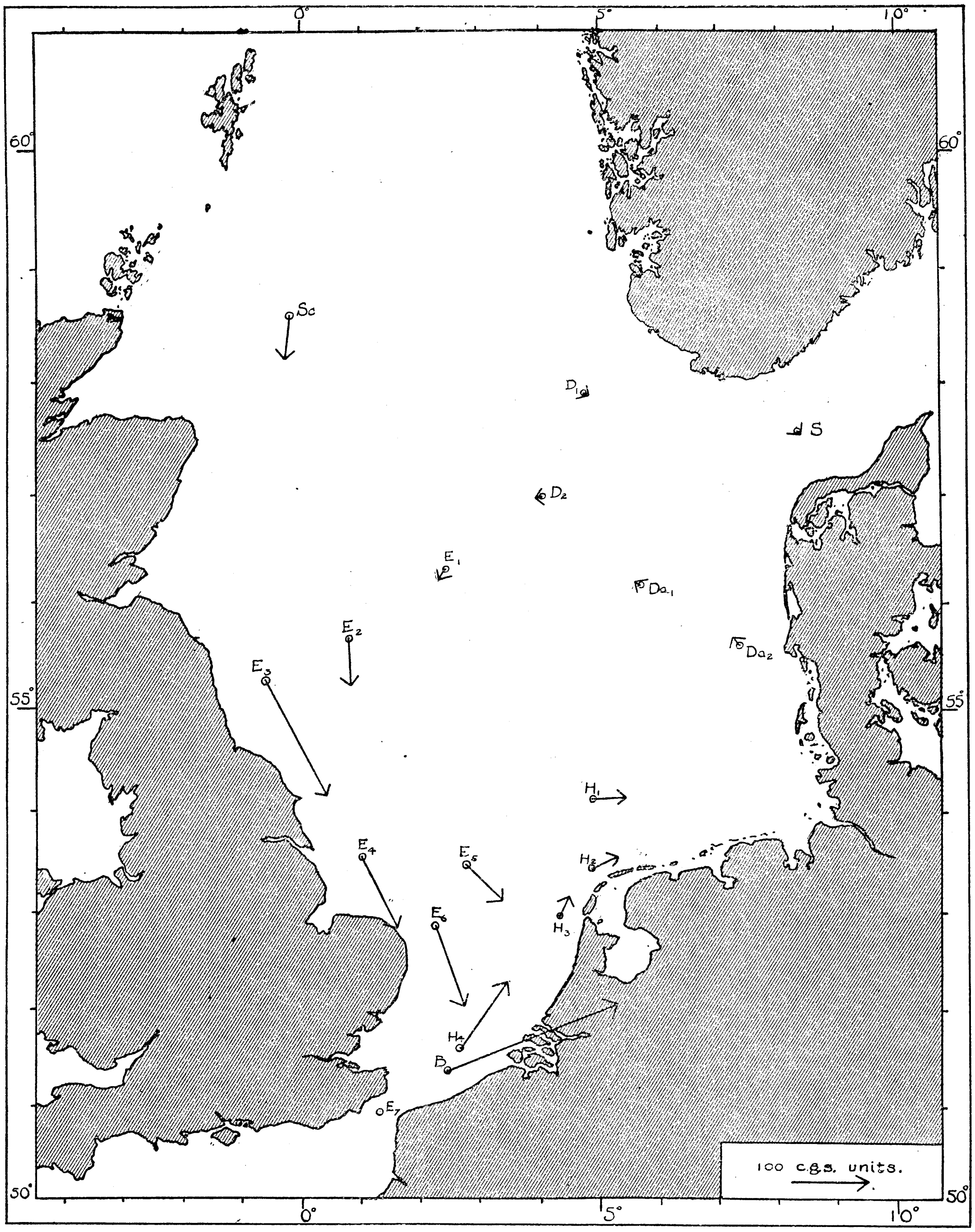

Fig. 18.—Mean flow of energy vector.

having been given to the data of Table I. and a little allowance having been made for Table III. data as in § 10. In the south-east region of the North Sea, preference has been given to data for places on the seaward side of the sandbanks and islands. Table X. and fig. 18 give the mean flow of energy vector in the North Sea in accordance with formula (18).

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TABLE I.—Values of Harmonic Constants (for M_2) for Coastal Elevations.

Station.	Latitude.	Longitude.	H.	γ.	Ref.
	° ′	° ′	cm.	°	
*Leith	N. 55 59	W. 3 10	181	55	1916
Hartlepool	54 41	1 11	157	98	1885
Sheerness	51 27	E. 0 45	192	359	1900
Ramsgate	51 20	1 25	187	341	1907
Dover	51 7	1 19	216	329	1913
Ostende	51 14	2 55	180	7	1889
Noord Hinder	51 35	2 37	120	7	1905
Weilingen	51 23	3 11	158	25	1910 S
Vlissingen	51 27	3 36	172	34	1910 S
Westkapelle	51 31	3 27	148	28	1910 S
Zierikzee	51 37	3 55	139	57	1910 S
Schouwen Bank	51 47	3 27	89	30	1905
Brouwershaven	51 44	3 55	115	59	1910 S
Hellevoetsluis	51 50	4 8	91	75	1910 S
Maas Light	51 55	4 14	72	68	1905
Scheveningen	52 7	4 16	75	73	1910 S
Katwyk	—	—	68	79	1910 S
Hook of Holland	51 58	4 6	77	63	1907
Ymuiden	52 28	4 34	67	104	1907
Haaks Light	52 59	4 18	67	171	1905
Helder	52 58	4 46	58	162	1907
Terschellingerbank	53 28	4 53	86	203	1905
Rothen Sande	53 51	8 5	122	322	1900
Heligoland	54 11	7 52	97	319	1900
Esbjerg	55 28	8 28	60	40	1906
Hirtshals	57 35	9 57	11	96	1906
Arendal	58 28	8 46	8	82	1906
Stavanger	58 58	5 45	15	271	1904 N
Bergen	60 24	5 19	44	287	1904 N

* Grade II ; remainder Grade I.

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TABLE II.—Current Data from "International Stations."

Station.	Name of station.	Lat. N.	Long. W.	Depth (metres).	Date and Ref.	u_0.	v_0.	γ'.	Direction of flood.
		° ′	° ′			cms. per sec.	cms. per sec.	°	°
Sc	—	58 28	0 12	122	1911	19·7	− 1·3	342	S. 9 E.
D_1	—	57 55	−4 45	100	1911	8·2	− 2·9	345	S. 3 E.
D_2	—	57 00	−4 0	ca. 60	1912	14·7	− 7·3	18	S. 47 W.
E_1	—	56 20	−2 23	ca. 70	1913	21·0	− 0·2	24	S. 30 W.
E_2	—	55 40	−0 45	81	1912	25·8	8·9	43	S. 23 W.
E_3	—	55 16	0 38	75	1911	32·2	7·0	92	S. 27 E.
E_4	Outer Dowsing	53 34	−0 59	ca. 20	1913	70·8	15·8	149	S. 22 E.
E_5	Swarte Bank	53 28	−2 46	ca. 25	1913	58·0	−28·8	223	S. 35 E.
E'_6	Smith's Knoll	52 52	−2 14	46·6	1911	70·0	− 6·6	252	S. 26 E.
,,	,,	,,	,,	,,	1912	89·5	− 7·4	249	S. 10 E.
E''_6	,,	52 44	−2 15	ca. 30	1913	84·9	−11·3	251	S. 4 E.
E_7	Varne	50 56	−1 17	29·3	1911	95·1	−14·3	346	N. 46 E.
,,	,,	,,	,,	,,	1912	86·2	− 6·7	334	N. 44 E.
,,	,,	,,	,,	,,	1913	79·7	−10·8	356	N. 20 E.
B	West Hinder	51 23	−2 27	32	1912	105·6	6·0	14	N. 67 E.
H_4	Noord Hinder	51 35	−2 37	32·4	1913	72·3	13·4	46	N. 27 E.
H_3	Haaks	52 58	−4 18	27	1913	52·4	− 4·4	115	N. 19 E.
H_2	Terschellingerbank	53 27	−4 52	27	1912	58·0	− 4·1	142	N. 59 E.
H_1	—	54 9	−4 51	42	1911	35·0	− 1·3	212	N. 87 E.
Da_2	Horn's Rev	55 34	−7 20	32	1911	38·9	− 3·9	100	N. 25 W.
,,	,,	,,	,,	,,	1912	33·8	− 0·5	58	N. 42 W.
,,	,,	,,	,,	,,	1913	38·7	− 0·4	76	N. 31 W.
Da_1	—	56 10	−5 37	55·5	1911	13·1	− 6·0	65	N. 63 W.
S	—	57 34	−8 25	106	1911	2·2	− 1·2	48	S. 67 E.
Sk	—	57 55	−9 42	80	1913	1·2	− 0·3	43	N. 14 E.

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TABLE III.—Approximate Values of Harmonic Constants (for M_2) for Coastal Elevations.

	Lat.	Long.	H.	γ.		Lat.	Long.	H.	γ.
	° ′	° ′	cms.	°		° ′	° ′	cms.	°
Balta	N. 60 45	W. 0 47	72	292	Boulogne . . .	N. 50 44	E. 1 35	296	325
Lerwick	60 9	1 8	62	319	C. Grisnez . .	50 52	1 35	253	324
Sumburgh Head	59 51	1 17	58	280	Calais	50 58	1 51	247	334
Scaddon	59 33	1 38	56	317	Gravelines . .	51 1	2 6	223	343
Otterswick . . .	59 17	2 30	123	266	Dunkerque . .	51 3	2 21	197	342
Kirkwall . . .	58 59	2 58	91	269	Nieuport . . .	51 10	2 44	187	357
Stromness . . .	58 57	3 18	111	250	*Ostende . . .	51 14	2 55	192	358
Duncansby Head	58 39	3 3	111	296	Blankenburghe	51 18	3 7	155	356
Wick	58 26	3 5	111	330	*Vlissingen . .	51 27	3 36	191	21
Banff	57 40	2 31	117	13	*Zierikzee . . .	51 37	3 55	145	59
Peterhead . . .	57 30	1 46	136	15	*Brouwershaven	51 44	3 55	123	49
Aberdeen . . .	56 9	2 5	155	28	*Hellevoetsluis	51 50	4 8	93	71
Stonehaven . .	56 58	2 12	166	33	*Hook of Holland	51 58	4 6	84	50
Montrose	56 42	2 28	163	66	*Ymuiden . .	52 28	4 34	84	74
Arbroath	56 33	2 35	165	46	*Helder . . .	52 58	4 46	58	149
Tay Bar	56 27	2 38	187	61	West Terschelling	53 21	5 13	70	236
Fifeness	56 17	2 35	181	63	Ameland Zeegat	53 29	5 31	73	255
*Leith	55 59	3 10	186	64	Borkum . . .	53 37	6 42	122	298
Dunbar	56 0	2 31	172	60	Nordeney . . .	53 43	7 10	123	314
Eyemouth . . .	55 52	2 5	181	64	Wangeroog W.	53 46	7 51	134	310
Berwick	55 47	2 0	177	66	*Rothen Sande .	53 51	8 5	137	315
Holy Island . .	55 40	1 47	177	71	*Heligoland . .	54 11	7 52	104	315
Alnmouth . . .	55 23	1 37	171	75	Elbe Is. Light .	54 1	8 14	113	346
Blyth	55 7	1 29	177	92	Cuxhaven . . .	53 54	8 40	123	2
Sunderland . .	54 55	1 21	171	95	Büsum	54 8	8 51	143	2
Seaham	54 50	1 19	171	96	Eider Light . .	54 16	8 16	142	326
*Hartlepool . .	54 41	1 11	166	94	Husum	54 29	9 4	162	35
Whitby	54 29	0 37	177	105	Pellworm . . .	54 31	8 42	134	31
Scarborough . .	54 17	0 23	181	117	Amrum	54 39	8 23	137	353
Filey	54 13	0 16	186	121	Wyk	54 41	8 35	116	29
Gridlington . . .	54 5	0 11	191	126	Lister Deep . .	55 4	8 10	87	351
Spurn Head . .	53 35	E. 0 7	223	153	Hjerting . . .	55 31	8 21	55	41
Grimsby	53 35	W. 0 4	230	157	Blaavandshuk .	55 33	8 5	61	13
Wells	52 57	E. 0 51	142	196	Nyminde Gab .	55 48	8 12	26	59
Blakeney Bar . .	52 59	0 59	178	181	Thyboron . . .	56 43	8 12	21	100
Yarmouth Roads	52 36	1 45	72	260	*Hirtshals . .	57 35	9 57	15	105
Lowestoft . . .	52 29	1 46	76	280					
Orfordness . . .	52 5	1 35	96	318					
Harwich	51 57	1 17	162	338	Christiansand .	58 9	8 0	12	108
The Nore . . .	51 29	0 49	187	8	Tananger . . .	58 56	5 35	18	267
*Sheerness . . .	51 27	0 45	206	5	*Stavanger . .	58 58	5 45	21	266
Margate	51 23	1 23	178	333	Skudesnaes . .	59 8	5 18	24	286
*Ramsgate . .	51 20	1 25	184	331	*Bergen . . .	60 24	5 19	49	288
Deal	51 13	1 25	186	319					
*Dover	51 7	1 19	230	320					
Folkstone . . .	51 5	1 11	233	315					
Dungeness . . .	50 54	0 58	253	305					

* These stations are also found in Table I.

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TABLE IV. (A–C).—Approximate Values of Harmonic Constants (for M_2) for Elevations at Sea.

TABLE IVA.—Admiralty.

N. Latitude.		E. Longitude.		Dates of observation.	H.	γ.
°	′	°	′		cms.	°
55	28	3	4	Oct. 18–20, 1913 . . .	41	121
55	2	3	3	Oct. 17–19, 1913 . . .	55	139
54	29	1	42	July 21–24, 1914 . . .	89	134
54	28	2	59	Oct. 18–19, 1913 . . .	78	145
53	58	4	0	Oct. 21–24, 1913 . . .	71	165
53	53	1	57	Oct. 22–23, 1913 . . .	91	158
53	37	4	14	Oct. 21–24, 1913 . . .	69	169
53	24	2	10	July 19–20, 1911 . . .	64	195
53	20	2	29	May 5–6, 1911 . . .	73	228
53	19	3	13	April 26–27, 1911 . . .	62	175
53	18	1	30	Aug. 9–11, 1913 . . .	117	175
53	17	1	38	July 17–21, 1913 . . .	117	179
53	9	2	0	May 8–9, 1911 . . .	96	195
53	3	2	53	May 4–5, 1911 . . .	41	206
52	58	1	59	Aug. 12–13, 1911 . . .	84	216
52	46	2	15	May 7–8, 1911 . . .	36	255
52	41	2	55	April 19–20, 1911 . . .	11	184
52	30	3	18	April 24–25, 1911 . . .	11	86
52	18	2	12	May 10–11, 1911 . . .	53	304
52	24	2	41	May 2–4, 1911 . . .	25	313
51	55	3	0	April 23–24, 1911 . . .	96	19

TABLE IVB.—Holzhauer (1886).

N. Latitude.		E. Longitude.		γ.
°	′	°	′	°
54	10	2	10	162
54	12	2	57	176
54	44	3	25	145 ?
55	3	5	55	297 ?
54	48	7	0	330 ?

TABLE IVC.—Merz (1921).

N. Latitude.		E. Longitude.		γ.
°	′	°	′	°
54	33	2	11	159
54	33	5	26	297
54	21	7	0	304
54	23	8	22	343
54	26	8	34	353

TABLE V.—Components of Vertical Mean Current, cms. per sec.

Station.	u_1.	u_2.	v_1.	v_2.
E_3	2·0	28·6	— 6·3	14·6
E_2	15·0	18·6	—13·2	— 1·0
E_1	15·9	8·8	— 9·1	— 5·4
Sc	18·7	— 5·7	2·6	— 1·9
D_2	10·8	— 3·6	— 9·7	— 6·7
D_1	8·0	— 1·9	— 0·6	— 2·8
Da_1	2·3	— 7·7	— 7·4	— 9·4
Da_2	— 4·4	—29·4	— 6·0	—18·4
H_2	23·4	—18·2	—43·8	24·0
H_1	2·3	— 0·1	—28·5	—17·7
E_4	—52·8	39·5	—52·8	1·4
E_5	—22·5	—45·1	—22·5	— 6·2
H_3	18·9	—45·9	18·9	13·8
S	— 0·2	1·3	1·7	1·2
Sk	0·3	0·7	0·7	0·5
E'_6	—28·0	—69·9	—14·5	—19·4
E_7	—63·6	20·7	51·4	— 5·6
H_4	—40·3	—50·4	31·7	15·5
B	—39·7	—10·7	96·9	10·4

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TABLE VI.—Components of Frictional Forces, c.g.s. units.

Station.	F_1.	F_2.	G_1.	G_2.
E_3	0·04	1·58	—0·21	0·81
E_2	0·72	0·79	—0·50	—0·14
E_1	0·57	0·32	—0·33	—0·19
Sc	0·63	—0·19	0·10	—0·05
D_2	0·28	—0·06	—0·27	—0·13
D_1	0·12	—0·02	—0·01	—0·03
Da_1	0·02	—0·16	—0·16	—0·23
Da_2	—0·30	—1·76	—0·29	—1·12
H_2	2·40	—1·65	—4·27	2·46
H_1	0·11	0·03	—1·63	—1·00
E_4	—6·58	4·53	—3·26	0·82
E_5	—2·66	—4·34	—3·67	—1·12
H_3	1·75	—4·06	—0·79	1·31
S	0·00	0·00	0·01	0·01
Sk	0·00	0·00	0·00	0·00
E'_6	—3·88	—9·32	—1·61	—2·74
E_7	—9·17	2·62	7·24	—1·30
H_4	—5·23	—6·00	3·46	2·37
B	—7·18	—1·42	8·67	8·28

TABLE VII.—Components of Surface-Gradients.

Station.	$10^7 \dfrac{\partial\zeta_1}{\partial x}$.	$10^7 \dfrac{\partial\zeta_1}{\partial y}$.	$10^7 \dfrac{\partial\zeta_2}{\partial x}$.	$10^7 \dfrac{\partial\zeta_2}{\partial y}$.
E_3	—48·8	—23·0	18·6	—45·1
E_2	—43·7	—16·4	19·3	—41·5
E_1	—24·6	—11·4	15·6	—23·6
Sc	10·9	—21·0	24·6	11·0
D_2	— 7·5	— 3·4	7·3	— 9·2
D_1	1·8	— 6·0	7·9	1·5
Da_1	1·9	10·9	— 8·1	— 0·7
Da_2	35·7	32·7	—23·3	31·1
H_2	—35·3	—46·2	1·4	—50·2
H_1	—34·4	26·4	—17·9	—38·2
E_4	—59·1	77·7	—97·0	—94·5
E_5	27·6	50·8	—21·9	1·1
H_3	46·4	—39·2	58·8	34·2
S	0·2	— 1·6	1·3	0·7
Sk	— 0·1	— 1·1	1·2	0·2
E'_6	91·4	64·3	—42·8	68·0
E_7	61·9	56·1	—106·8	54·4
H_4	125·8	13·7	—20·6	96·7
B	150·9	3·6	—40·2	125·1

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TABLE VIII.—Summary of Information obtained along Sectional Lines.

Line.	L.	λ.	ζ_1.	ζ_2.	$10^7\frac{\partial\zeta_1}{\partial x}$.	$10^7\frac{\partial\zeta_1}{\partial y}$.	$10^7\frac{\partial\zeta_2}{\partial x}$.	$10^7\frac{\partial\zeta_2}{\partial y}$.	$10^7\frac{\partial\gamma}{\partial x}$.	$10^7\frac{\partial\gamma}{\partial y}$.	$10^7\frac{\partial H}{\partial x}$.	$10^7\frac{\partial H}{\partial y}$.	H.	γ.	ψ.
	°	°	cms.	cms.									cms.	°	°
I.	1	55·16	— 14	150	—48	—25	18	—45	0·30	0·19	22	—42	151	95	302
	0	55·44	— 13	116	—48	—19	20	—44	0·39	0·20	25	—42	117	96	297
	—1	55·76	— 7	82	—41	—15	19	—39	0·48	0·22	23	—38	82	95	295
	—2	56·15	— 1	53	—29	—12	16	—28	0·54	0·24	17	—28	53	91	294
	—3	56·59	4	32	—18	— 8	12	—16	0·61	0·19	10	—17	32	83	287
	—4	57·00	6	20	— 7	— 3	7	— 9	0·41	0·01	5	— 9	21	81	271
II.	—3	56·64	4	32	—17	— 9	13	—15	0·58	0·22	11	—16	32	83	291
	—4	57·26	5	21	— 6	— 7	10	— 4	0·36	0·26	8	— 5	22	77	306
	—5	58·20	1	16	4	— 6	7	4	—0·22	0·39	7	4	16	86	29
III.	1	58·60	56	— 8	23	—22	24	27	0·47	0·41	19	—25	57	352	311
	0	58·42	47	7	8	—21	25	8	0·49	0·23	11	—19	48	8	295
	—1	58·17	36	14	— 2	—18	24	— 3	0·58	0·09	7	—18	39	21	279
	—2	57·83	25	19	— 7	—15	21	— 8	0·68	0·09	7	—17	31	37	278
	—3	57·45	14	20	— 9	—10	15	—10	0·68	0·10	7	—14	24	55	278
	—5	56·48	2	17	— 3	4	— 1	— 5	0·17	—0·27	— 1	— 5	17	83	212
	—6	56·00	8	14	7	15	—13	5	—0·79	—0·66	— 8	12	16	60	130
	—7	55·66	28	15	28	29	—22	25	—1·01	0·26	14	37	32	28	76
IV.	—4·87	53·45	— 79	—34	—35	—46	1	—50	—0·17	0·32	32	62	86	203	28
	,,	54·0	— 57	—30	—35	21	—14	—40	—0·07	0·67	36	0	66	208	6
	,,	54·5	— 39	—21	—32	30	—20	—33	0·06	0·99	38	—11	44	208	357
	,,	55·0	— 24	— 8	—25	28	—23	—22	0·56	1·20	31	—20	25	198	335
	,,	55·2	— 18	— 4	—22	25	—22	—19	0·95	1·36	27	—21	18	193	325
	,,	55·4	— 13	0	—20	22	—21	—16	1·62	1·23	20	—22	13	180	307
	,,	56·0	— 2	12	—11	15	—13	—10	1·10	—1·11	—11	—13	12	99	225
	,,	56·2	0	15	— 8	12	—10	— 9	0·53	—0·80	—10	— 9	15	90	213
	,,	57·0	4	19	0	4	0	— 4	0·00	—0·26	0	— 3	19	78	180
V.	—1	55·67	— 12	84	—43	—15	15	—41	0·48	0·24	21	—39	85	98	297
	—2	,,	— 17	60	—40	— 6	1	—37	0·60	0·26	12	—34	62	106	293
	—3	,,	— 19	38	—34	1	— 9	—32	0·83	0·33	7	—29	42	117	292
	—4	,,	— 15	19	—27	11	—16	—24	1·31	0·26	4	—26	24	132	281
	—5	,,	— 6	6	—15	21	—20	—11	3·26	—0·94	— 4	—24	8	135	254
	—6	,,	9	6	2	27	—21	6	—1·67	—0·89	—10	25	10	34	118
VI.	—0·98	53·57	—156	21	—59	78	—97	—94	0·58	0·46	46	—90	157	172	308
	—2·00	53·55	—114	—20	—	—	—	—	—	—	—	—	116	190	—
	—2·77	53·47	— 90	—34	28	51	—22	1	0·32	0·18	—19	—48	96	201	299
	—3·50	53·30	— 72	—24	—	—	—	—	—	—	—	—	76	198	—
	—4·30	52·97	— 66	10	46	—39	59	34	—0·97	—0·41	—37	43	67	171	113

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TABLE IX.—Harmonic Elevation-constants at the International Stations.

Station.	H.	γ.	Station.	H.	γ.
	cms.	°		cms.	°
Sc	49	5	B	150	358
D_1	17	83	H_4	120	7
D_2	21	74	H_3	67	171
E_1	43	87	H_2	86	203
E_2	90	96	H_1	59	208
E_3	142	95	Da_2	42	25
E_4	157	172	Da_1	16	72
E_5	96	201	S	12	86
E'_6	68	223	Sk	11	94
E_7	245	—			

TABLE X.—Mean Flow of Energy Vector.

Station.	10^{-8} E.	θ.	Station.	10^{-8} E.	θ.
		°			°
Sc	57	353	B	242	112
D_1	2·7	74	H_4	109	145
D_2	6·4	277	H_3	29	154
E_1	13·7	329	H_2	35	114
E_2	62	2	H_1	44	93
E_3	171	28	Da_2	15	216
E_4	103	27	Da_1	6·0	240
E_5	66	46	S	1·2	45
E'_6	113	21	Sk	0·3	148
E_7	—	—			

E = magnitude of vector, c.g.s. units.
θ = direction of vector measured from the south round by the east.

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